因果的に無矛盾な散逸流体の現象論

— 解析・数値的に扱いやすそうな EIT の簡単化 —

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BH 磁気圏勉強会：at 大同大学, 2011 2/28, 3/1, 3/2

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The Ins & Outs of BHs : at Annapolis, USA, 15-17.Nov.2010

なんとなく日本語で書きなおそうと
思ったけど間に合いませんでした。

Plan of talk

1. Introduction – beyond the Naiver-Stokes and Fourier laws –

2. Causally Consistent Dissipative Hydrodynamics

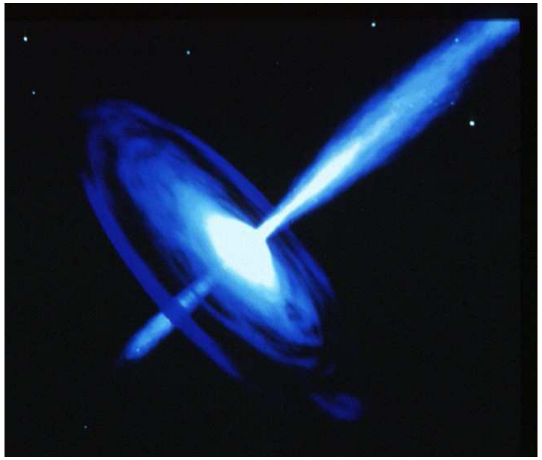
$\rightarrow$ **Extended Irreversible Thermodynamics (EIT)**

3. How to utilize the EIT : **A proposal to make use of EIT**

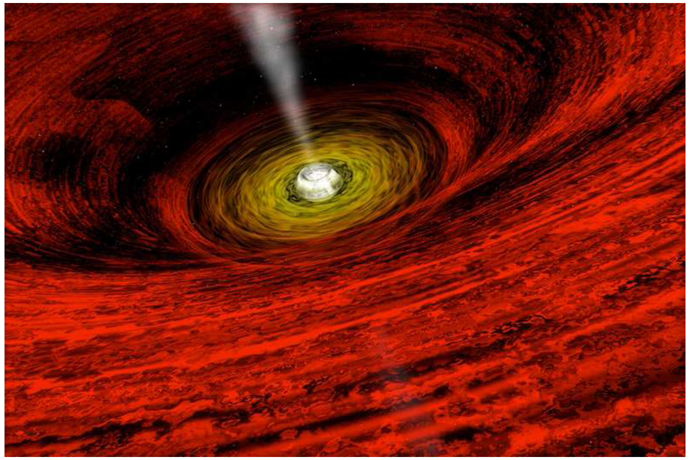
4. Exercise : Dissipative perturbation of hydrostatic star

5. Toward the BH accretion : Present status of research

6. Summary and Next Task



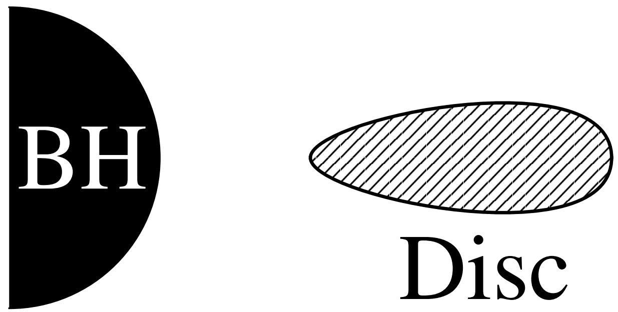
— let's get closer $\rightarrow$



existing theory: Far from BH $\Longrightarrow$ this study : "Near BH Region"

1. Introduction

1.1 Motivations



(1) I want to "SEE" black holes !

→ **BH Shadow** is interesting

↳ Observational Evidence for the Existence of BH Horizon
Direct Test of Gen. Rel. with Strong Gravity
BH Shadow can be a good partner of Grav. Wave

(2) There is a rather large **controversy** about obs. of BH candidates

→ due to large **dependence on accretion model**:
- Dissipative process
- Radiative process
- and so on ...

- For Detail of BH Shadow and Resolution of the controversy

→ **Theory of accretion with less model dependence is needed**

1.2 Accretion Flow (ex. Accretion Disk) – Dissipations –

- In Accretion Disk : Grav. Potential $\xrightarrow{(1)}$ Thermal Energy $\xrightarrow{(2)}$ Rad. Power

  - (1) : differential rotation + viscosity = frictional heat
    → **Dissipations (Heat & Viscosity) are essential**
  - (2) : Temperature distribution → Multi-Temperature Thermal Spectrum

→ Note : **Gen. Rel. effect has been neglected so far**

⇑ technical reason (Accretion Disk far from BH)

- Viscosity : **Navier-Stokes**
- Heat Flow : **Fourier Law**

violate the causality !

→ Including GR effects may reduce the variety of accretion model near BH

Aim : General Relativistic Formulation and Analysis of Dissipative Accretion Flow/Disk

(In future : Details of BH Shadow and Direct Test of GR are interesting)

2. Extended Irreversible Thermodynamics — EIT —

(→ includes Israel's theory of causal dissipative hydrodynamics)

2.1 Limit of local "equilibrium" idea – Navier-Stokes & Fourier –

• with the Idea of **Local Equilibrium** : time-differential ∂_t is absent

$$
\begin{cases}
\text{Heat Flow} & : \vec{q} = -\lambda \vec{\nabla} T \quad \rightarrow \text{Fourier law of heat conduction} \\
\text{Bulk Viscosity} & : \Pi = -\zeta \vec{\nabla} \cdot \vec{v} \\
\text{Shear Viscosity} & : \overset{\circ}{\Pi}{}^{ij} = -2\,\eta\,\overset{\circ}{v}{}^{ij} \;,\; \overset{\circ}{v}{}^{ij} := \frac{1}{2}\Big(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\Big) - \frac{1}{3}\delta^{ij}\,\vec{\nabla}\cdot\vec{v} \\
& \rightarrow \Pi \;,\; \overset{\circ}{\Pi}_{ij} : \text{Navier-Stokes eq. of Viscous Fluid}
\end{cases}
$$

→ "Relaxation time of non-stationary dissipation to stationary one" = 0

→ No Retarded Effect of Dissipations ··· Infinitely Fast Propagation !

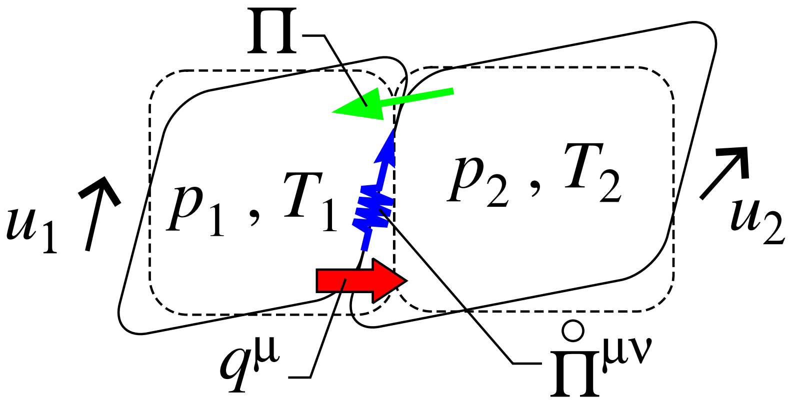
→ { **Short Periodic Phenomena can't be treated** (even in non-rel.)
  Causality is broken in relativistic case

2.2 Basic assumptions on "non-equilibrium thermodynamics"

- Assumption 1 ($\rightarrow$ it is rather the "Fact" !)

  Local equilibrium idea can not describe causal dissipative phenomena:

Assumption 1 : The dissipative fluid is in Local Non-equilibrium states.



$p_i = p_{\rm ne}(x_i)$ ($i = 1, 2$)

$T_i = T_{\rm ne}(x_i)$

$p_{\rm ne}$: non-equilibrium pressure

$T_{\rm ne}$: non-equilibrium temperature

q^μ : heat flux (vector)

Π : bulk viscosity (scalar)

$\mathring{\Pi}^{\mu\nu}$: shear viscosity (tensor)

u^μ : fluid velocity

These quantities are functions of spacetime coordinates $x^\mu = (t, \vec{r})$

- Assumption 2

  To formulate "non-equilibrium thermodynamics" for fluid elements, we need state variables suitable for characterizing non-equilibrium states:

Assumption 2 : Non-equilibrium state variables are distinguished into:

Non-equilibrium Vestiges （ vestige：痕跡，形跡，名残り） :

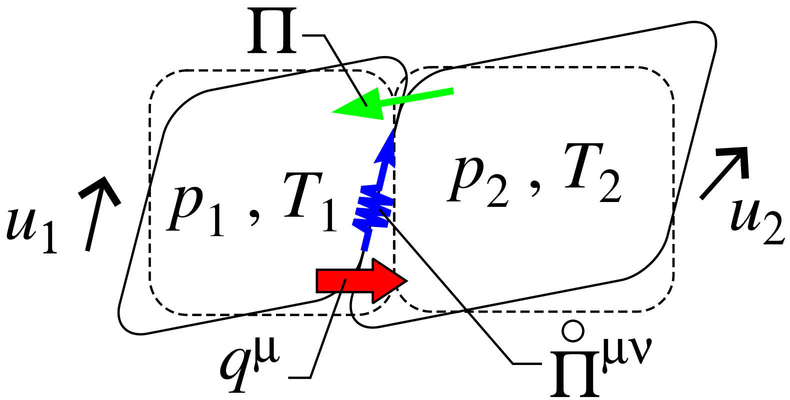
State variables appearing in ordinary equilibrium thermodynamics

$\to$ This can remain finite value at local equilibrium limit.

Dissipative Fluxes （散逸流量） :

State variables which must vanish at local equilibrium limit.

$\to$ e.g. heat flux, bulk viscosity, shear viscosity, etc...



$p_{\rm ne}$, $T_{\rm ne}$: non-equilibrium vestiges

q^μ , Π , $\mathring{\Pi}^{\mu\nu}$: dissipative fluxes

(u^μ : the *dynamical* variable)

- Assumption 3

  How many state variables are independent ?

  → Hereafter consider the **Simple Fluid**, for simplicity

  ◇ if it is in thermal equilibrium:

  $$\text{No. of Indep. state variables} = \begin{cases} 2 & \text{for closed system} \\ 3 & \text{for open system} \end{cases}$$

  ◇ in Navier-Stokes and Fourier laws ⋯ Not causal !

  Dissip. Fluxes = "function of u^μ and equilibrium state variables"

  → Minimal correction :

Assumption 3 :

$$\text{No. of Indep. Non-equilibrium Vestiges} = \begin{cases} 2 & \text{for closed system} \\ 3 & \text{for open system} \end{cases}$$

$$\text{No. of Indep. Dissipative Fluxes} = 3 \quad (\rightarrow \text{e.g. } q^\mu , \Pi , \mathring{\Pi}^{\mu\nu})$$

→ Implies the existence of Non-equilibrium equations of states, and ⋯

◇ A result of assumptions 2 and 3

→ Remember ; Thermodynamic conjugate state variable

e.g. "Conjugate variable to entropy S " $:= \dfrac{\partial E(S,V)}{\partial S} = T$

→ "Thermodynamic conjugates to q^{μ} , Π , $\overset{\circ}{\Pi}{}^{\mu\nu}$ " are also

the "Dissipative Fluxes"

→ Assumptions 2 and 3 imply

$$\frac{\partial[\text{"dependent" state variable}]}{\partial[\text{indep. dissip. flux}]} \to 0 \quad \text{as} \quad [\text{indep. dissipative fluxes}] \to 0$$

---

e.g. $$\frac{\partial S_{\mathrm{ne}}(p_{\mathrm{ne}}\,,\,T_{\mathrm{ne}}\,,\,q^{\mu}\,,\,\Pi\,,\,\overset{\circ}{\Pi}{}^{\mu\nu})}{\partial \Pi} \to 0 \quad \text{as} \quad (q^{\mu}\,,\,\Pi\,,\,\overset{\circ}{\Pi}{}^{\mu\nu}) \to 0$$

- Assumption 4

Assumptions 1, 2 and 3 are the "non-equilibrium 0th law"

($\cdots$ existence and basic property of non-equilibrium states)

$\rightarrow$ Guiding principle for obtaining "evolution eqs. of Dissipative Fluxes"

should be separately specified

$\rightarrow$ To do so, 2 points are important:

$\diamond$ Define ; Self-production rate of entropy , $\sigma_{\rm s} \;:=\; S_{\rm ne}{}^{\mu}{}_{;\,\mu}$

where $S_{\rm ne}{}^{\mu}$: Non-equilibrium entropy current $\cdots$ shown later

$\diamond$ A fact in Local Equilibrium laws (e.g. Fourier law) ;

$$\sigma_{\rm s(LE)} \;=\; \underline{\vec{q}\cdot(-\vec{\nabla}T)} \;=\; \lambda^{-1}q^2 > 0 \;\text{ if } \lambda > 0 \;\rightarrow\; \textbf{2nd law}$$

= [Dissipative Flux] $\times$ [**Thermodynamic Force**]

"Bilinear form" of entropy production rate

$\rightarrow$ Simply extend to non-equilibrium states $\cdots$

Assumption 4 : For self-entropy production rate, $\sigma_{\rm s} := S_{\rm ne}{}^{\mu}{}_{;\mu}$, assume:

(4-a) Non-negative production rate, $\underline{\sigma_{\rm s} \geq 0 \quad \text{(2nd law)}}$

(4-b) $\sigma_{\rm s}$ is expressed by the "Bilinear Form",
where "non-equilibrium thermodynamic force" should reduce to
non-relativistic "causal" phenomenology at non-relativistic limit
with neglecting dissipative interaction (e.g. heating by viscous flow)

◇ **Maxwell-Cattaneo laws** : "naive causal extension" of acausal laws

$$\begin{cases} \text{Heat flux} \ : \ \tau_{\rm h}\,[d\vec{q}/dt] + \vec{q} \ = \ -\lambda\,\vec{\nabla}T \\ \text{Bulk Vis.} \ : \ \tau_{\rm b}\,[d\Pi/dt] + \Pi \ = \ -\zeta\,\vec{\nabla}\cdot\vec{v} \\ \text{Shear Vis.} \ : \ \tau_{\rm s}\,[d\mathring{\Pi}^{ij}/dt] + \mathring{\Pi}^{ij} \ = \ -2\,\eta\,\mathring{v}^{ij} \end{cases}$$

τ's : Relaxation times of Dissipative Fluxes

→ Note : These with local non-equilibrium idea can NOT guarantee 2nd law,

$$\sigma_{\rm s(LE)} = \vec{q}\cdot(-\vec{\nabla}T) = \frac{\tau_{\rm h}}{\lambda}\frac{{\rm d}q^2}{{\rm d}t} + \frac{q^2}{\lambda}\cdots ??? \ \rightarrow \ \text{Form of } \sigma_{\rm s} \text{ is important !}$$

• Condition 1

The problem is how to determine $S^{\mu}_{\rm ne}$

→ Physically consistent $S^{\mu}_{\rm ne}$ is already known under the condition:

Condition 1 : Consider only "weak" dissipative fluxes which validate the 2nd Order Approximation of E.O.S. : (example by specific entropy, $s_{\rm ne}$)

$$s_{\rm ne}(p_{\rm ne}\,,\,T_{\rm ne}\,,\,{\rm DF}) \\ = s_{\rm eq}(p_{\rm eq}\,,\,T_{\rm eq}\,,\,{\rm DF}) + [\text{2nd order of Dissipative Fluxes (DF)}]$$

(let us consider only weak non-equilibrium states)

◇ 1st order term of DF vanishes due to assumptions 2 and 3

◇ $T_{\rm eq}$: temperature of **Fiducial Equilibrium State**

→ Equilibrium State of Imaginary Perfect Fluid's element with $\rho_{\rm ne}$ and u^{μ}

(ρ : rest mass density)

→ Apply the condition 1 to $S^{\mu}_{\rm ne}$ $\cdots$

→ Non-equilibrium Entropy Current under the condition 1 :

$$S_{\rm ne}{}^{\mu} := \rho_{\rm ne}\, s_{\rm ne}\, u^{\mu} + \frac{1}{T}\, q^{\mu} + \beta_{\rm hb}\, \Pi\, q^{\mu} + \beta_{\rm hs}\, q_{\alpha}\, \overset{\circ}{\Pi}{}^{\alpha\mu}$$

where $\begin{cases} \beta_{\rm hb} : \text{ Interaction coefficient between } q^{\mu} \text{ and } \Pi \\ \beta_{\rm hs} : \text{ Interaction coefficient between } q^{\mu} \text{ and } \overset{\circ}{\Pi}{}^{\mu\nu} \end{cases}$

◇ By Assumption (4-b), $\sigma_{\rm s} = S_{\rm ne}{}^{\mu}{}_{;\mu} =: q_{\mu}\, X_{\rm h}^{\mu} + \Pi\, X_{\rm b} + \overset{\circ}{\Pi}_{\mu\nu}\, X_{\rm s}^{\mu\nu}$

(X's : "Non-equilibrium Thermodynamic Forces")

→ By Assumption (4-a) + Condition 1 for $\sigma_{\rm s}$,

$$X_{\rm h}^{\mu} = b_{\rm h}\, q^{\mu} \;,\; X_{\rm b} = b_{\rm b}\, \Pi \;,\; X_{\rm s}^{\mu\nu} = b_{\rm s}\, \overset{\circ}{\Pi}{}^{\mu\nu} \;\; (\, b > 0 \,) \;\; \cdots \;\; (*)$$

→ By Assumption (4-b) with Maxwell-Cattaneo laws, b's are determined

→ Eq.($*$) are re-arranged to **Evolution eqs. of Dissipative Fluxes**

2.3 Basic quantities of EIT (other than spacetime metric $g_{\mu\nu}$)

- **Fiducial equilibrium state variables** (0th order in E.O.S.) :

  ex.) Temperature : T , Specific Internal Energy : ε [erg/g]

  Pressure : p , Rest Mass Density : $\rho\,(=V^{-1})$

  → Omit the suffix "eq" for simplicity

- **Dissipative Fluxes** (responsible to non-equilibrium irreversibility) :

  Heat Flux : q^{μ} , Bulk Viscosity : Π , Shear Viscosity : $\overset{\circ}{\Pi}{}^{\mu\nu}$

- **Fluid 4-velocity** (dynamical variable of fluid) : $u^{\mu}(\rho\,,\,T\,,\,q^{\alpha}\,,\,\Pi\,,\,\overset{\circ}{\Pi}{}^{\alpha\beta})$

  → **Remark :** u^{μ} is $\begin{cases}\text{determined by the basic eqs. of EIT (see next)}\\ \text{not perturbed by dissipative fluxes in Condition 1}\end{cases}$

- Energy-Momentum tensor, $T^{\mu\nu} := \rho\,\varepsilon\,u^{\mu} + 2\,u^{(\mu}q^{\nu)} + (p+\Pi)\,\Delta^{\mu\nu} + \overset{\circ}{\Pi}{}^{\mu\nu}$

  ($\Delta^{\mu\nu}$: projection tensor → see next)

2.4 Basic Eqs. of EIT – Gen. Rel. version – (includes Israel's theory)

- **E.O.S. for "fiducial" equilibrium state** : $p = p(\rho, T)$, $\varepsilon = \varepsilon(\rho, T)$

- **Constraints** : $u^\alpha u_\alpha = -1$, $q^\alpha u_\alpha = 0$, $\mathring{\Pi}^{\mu\alpha} u_\alpha = 0$, $\mathring{\Pi}^{\alpha}{}_{\alpha} = 0$

- Current Conservation $(\rho u^\mu)_{;\mu} = 0$, Energy-Momentum Cons. $T^{\mu\nu}{}_{;\nu} = 0$:

Mass Cons.: $\dot{\rho} + \rho u^\alpha{}_{;\alpha} = 0$

Energy Cons.: $\rho \left(\dot{\varepsilon} + p \dot{V} \right) = -q^\alpha{}_{;\alpha} - q^\alpha \dot{u}_\alpha - \left(\Pi \Delta^{\alpha\beta} + \mathring{\Pi}^{\alpha\beta} \right) u_{\alpha;\beta}$

EOM : $(\rho\varepsilon + p + \Pi)\, \dot{u}^\mu = -\dot{q}^\mu + q_\alpha \dot{u}^\alpha u^\mu - u^\alpha{}_{;\alpha} q^\mu - q^\alpha u^\mu{}_{;\alpha}$

(light speed $c = 1$) $\qquad - \Delta^{\mu\alpha} \left[(p + \Pi)_{,\alpha} + \mathring{\Pi}_{\alpha}{}^{\beta}{}_{;\beta} \right]$

where
$$\begin{cases} \dot{\mathbf{Q}} := u^\alpha \mathbf{Q}_{;\alpha} & : \text{4-dim. Lagrange derivative} \\ \Delta_{\mu\nu} := u_\mu u_\nu + g_{\mu\nu} & : \text{Vertical Projection about } u^\mu \end{cases}$$

- Assumption 4 + Condition 1 for $S_{\rm ne}^{\mu}$ $\rightarrow$ Evolution of Dissipations

$$\textbf{Heat:}\ \tau_{\rm h}\,\overset{\bullet}{q}{}^{\mu} \ = \ -\left[1+\lambda\,T^2\left(\frac{\tau_{\rm h}}{2\lambda T^2}\,u^{\nu}\right)_{;\nu}\right]q^{\mu} - \lambda\,T\,\overset{\bullet}{u}{}^{\mu} + \tau_{\rm h}(q^{\nu}\overset{\bullet}{u}_{\nu})\,u^{\mu}$$

$$-\,\lambda\,\Delta^{\mu\nu}\left[T_{,\nu} - T^2\left\{\beta_{\rm hb}\,\Pi_{,\nu} + (1-\gamma_{\rm hb})\,\Pi\,\beta_{{\rm bh}\,,\nu}\right.\right.$$

$$\left.\left. +\,\beta_{\rm hs}\,\overset{\circ}{\Pi}{}_{\nu}{}^{\alpha}{}_{;\alpha} + (1-\gamma_{\rm hs})\,\beta_{{\rm hs}\,,\alpha}\,\overset{\circ}{\Pi}{}^{\alpha}{}_{\nu}\right\}\ \right]$$

$$\textbf{Bulk Vis.:}\ \tau_{\rm b}\,\overset{\bullet}{\Pi} \ = \ -\left[1+\zeta\,T\,\left(\frac{\tau_{\rm b}}{2\zeta T}\,u^{\mu}\right)_{;\mu}\right]\Pi - \zeta\,u^{\mu}{}_{;\mu}$$

$$+\,\zeta\,T\,\left(\beta_{\rm hb}\,q^{\mu}{}_{;\mu} + \gamma_{\rm hb}\,q^{\mu}\,\beta_{{\rm hb}\,,\mu}\right)$$

$$\textbf{Shear Vis.:}\tau_{\rm s}\,(\overset{\circ}{\Pi}{}^{\mu\nu})^{\bullet} \ = \ -\left[1+2\,\eta\,T\,\left(\frac{\tau_{\rm s}}{4\eta T}\,u^{\alpha}\right)_{;\alpha}\right]\overset{\circ}{\Pi}{}^{\mu\nu} + 2\,\tau_{\rm s}\,\overset{\bullet}{u}{}^{\alpha}\,\overset{\circ}{\Pi}{}_{\alpha\,(\mu}\,u_{\nu)}$$

$$-\,2\,\eta\,\left[\,u_{\mu;\nu} - T\,\left\{\beta_{\rm hs}\,q_{\mu;\nu} + \gamma_{\rm hs}\,\beta_{{\rm hs}\,,\mu}\,q_{\nu}\right\}\,\right]^{\circ}$$

last term in Shear Vis. : $\left[A_{\mu\nu}\right]^{\circ} := \Delta_{\mu}{}^{\alpha}\,\Delta_{\nu}{}^{\beta}\,A_{(\alpha\beta)} - \dfrac{1}{3}\,\Delta_{\mu\nu}\,\Delta^{\alpha\beta}A_{\alpha\beta}$

3. How to utilize the EIT – perturb u^μ around perfect fluid –

3.1 Basic consideration

- Sec.2 : Thermodynamic variables (EOS) are perturbed by dissipations,

  but dynamical variable u^μ is NOT

→ Tacit understanding in EIT (also in Navier-Stokes) :

A dissipative flow which is essentially different from that of perfect fluid ("far" from perfect fluid's flow) is possible even with weak dissipations.

→ Basic eqs. of EIT look very complicated and are difficult to solve.

⇓ then ...

Consider the case that u^μ is "near" a perfect fluid flow:

$$u^\mu = u^\mu_{(\mathrm{p})} + \delta u^\mu + O(\mathrm{dissipation})^2 \quad , \quad \begin{cases} u^\mu_{(\mathrm{p})} & : \text{background perfect flow} \\ \delta u^\mu & : \text{1st order of dissipation} \end{cases}$$

3.2 Zero-th and 1st order eqs.

- Assumption of the order of parameters:

$$\begin{cases} \underline{\text{Transport coeff.}} & : \ \lambda\,,\ \zeta\,,\ \eta \ = \ O(\text{dissipation}) \ < \ O(1) \\ \underline{\text{Relaxation time}} & : \ \tau\text{'s} \ \sim \ O(1) \\ \underline{\text{Interaction coeff.}} & : \ \beta\text{'s}\,,\ \gamma\text{'s} \ \sim \ O(1) \end{cases}$$

(Relaxation time and Interaction coeff.) $\mapsto$ leave these rather free

$\to$ Dissipations are small

But, once dissipations arise, they can interact within the duration τ's

- **0th order eqs.** from EIT's basic eqs. $\to$ Background perfect fluid flow

$$\begin{cases} \rho' + \rho\, u^{\alpha}_{(\mathrm{p})\,;\alpha} \ = \ 0 \\ \rho\,(\,\varepsilon' + p\,V'\,) \ = \ 0 \\ (\,\rho\,\varepsilon + p\,)\,\left(u^{\mu}_{(\mathrm{p})}\right)' \ = \ -p_{,\alpha}\,\Delta^{\alpha\mu}_{(\mathrm{p})} \end{cases}$$

, where $\mathbf{Q}' := u^{\alpha}_{(\mathrm{p})}\,\mathbf{Q}_{;\alpha}$

- **1st order eqs.** – Evolution of q^μ , Π , $\mathring{\Pi}^{\mu\nu}$, δu^μ

$$
\left\{
\begin{array}{lcl}
\tau_{\rm h}(q^\mu)' & = & -\left[1+\lambda T^2\left(\dfrac{\tau_{\rm h}}{2\,\lambda\,T^2}\,u^\alpha_{\rm (p)}\right)_{;\alpha}\right] q^\mu - \lambda\,T\,(u^\mu_{\rm (p)})' \\
 & & \qquad + \tau_{\rm h}\,(\,q^\alpha\,u_{{\rm (p)}\,\alpha}\,)\,u^\mu_{\rm (p)} - \lambda\,T_{,\alpha}\,\Delta^{\alpha\mu} \\
\tau_{\rm b}\Pi' & = & -\left[1+\zeta\,T\left(\dfrac{\tau_{\rm b}}{2\,\zeta\,T}\,u^\alpha_{\rm (p)}\right)_{;\alpha}\right]\Pi - \zeta\,u^\alpha_{{\rm (p)};\alpha} \\
\tau_{\rm s}(\mathring{\Pi}^{\mu\nu})' & = & -\left[1+2\,\eta\,T\left(\dfrac{\tau_{\rm s}}{4\,\eta\,T}\,u^\alpha_{\rm (p)}\right)_{;\alpha}\right]\mathring{\Pi}^{\mu\nu} + 2\,\tau_{\rm s}\,u'_{{\rm (p)}\alpha}\,\mathring{\Pi}^{\alpha(\mu}\,u^{\nu)}_{\rm (p)} \\
 & & \qquad - 2\,\eta\,[\,u^{\mu;\nu}_{\rm (p)}\,]^\circ \\
\hline
(\,\rho\,\varepsilon+p\,)\,(\delta u^\mu)' & = & -(\,\rho\,\varepsilon+p\,)\,\delta u^\alpha\,u^\mu_{{\rm (p)};\alpha} - \Pi\,(u^\mu_{\rm (p)})' - (q^\mu)' \\
 & & \quad +(\,q^\alpha\,u_{{\rm (p)}\,\alpha}\,)\,u^\mu_{\rm (p)} - u^\alpha_{{\rm (p)};\alpha}\,q^\mu - q^\alpha\,u^\mu_{{\rm (p)};\alpha} \\
 & & \quad -(\,\Pi_{,\alpha} + \mathring{\Pi}_\alpha{}^\beta{}_{;\beta}\,)\,\Delta^{\alpha\mu}_{\rm (p)} - p_{,\alpha}\,\delta\Delta^{\alpha\mu}
\end{array}
\right.
$$

- **1st order eqs.** – Constraints

$$\begin{cases} \delta u^{\alpha}\,\rho_{,\alpha} + \rho\,\delta u^{\alpha}{}_{;\alpha} \;=\; 0 \\ \rho\,(\varepsilon_{,\alpha} + p\,V_{,\alpha})\,\delta u^{\alpha} + \Pi\,\rho\,V' + q^{\alpha}{}_{;\alpha} + q^{\alpha}\,u_{(\mathrm{p});\alpha} + \overset{\circ}{\Pi}{}^{\alpha\beta}\,u_{(\mathrm{p})\alpha;\beta} \;=\; 0 \end{cases}$$

$\to$ These constraints restrict the "arbitrary functions"

which arise in solving Evolution eqs.

- Notes for 1st order eqs. :

◇ Interactions among dissipative fluxes (β's and γ's) disappear at 1st order

$\to$ The interactions are higher order effects in perturbation of u^{μ}

◇ Evolution eqs. of q^{μ} , Π , $\overset{\circ}{\Pi}{}^{\mu\nu}$ are independent of each other

$\to$ δu^{μ} is solved after solving dissipative fluxes (q^{μ} , Π , $\overset{\circ}{\Pi}{}^{\mu\nu}$)

4. Exercise : dissipative perturbation of hydrostatic star

4.1 Zero-th order perfect fluid : Hydrostatic star by TOV eqs.

- Metric : $ds^2 = -A(r)dt^2 + \dfrac{dr^2}{B(r)} + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$\text{where} \begin{cases} \dfrac{dA}{dr} = \dfrac{2A}{r^2 B(r)}\,[\,M(r) + 4\pi\, r^3\, p(r)\,] \\[2ex] B = 1 - \dfrac{2\,M(r)}{r} \quad , \quad M(r) = \displaystyle\int_0^r dr\, 4\pi\, r^2\, \rho(r) \end{cases}$$

- 0th order perfect fluid :

$$\begin{cases} \text{4 velocity} & : u^\mu_{(\mathrm{p})} = \left(1/\sqrt{A}\,,\, 0\,,\, 0\,,\, 0\right) \\[1ex] \text{Balance eq.} & : \dfrac{dp(r)}{dr} = -\dfrac{(dA/dr)}{2A}\,[\,\rho\,\varepsilon + p\,] \\[1ex] \text{E.O.S.} & : p = p(\rho, T)\ ,\ \varepsilon = \varepsilon(\rho, T)\ \cdots\ \text{leave it arbitrary} \end{cases}$$

4.2 General solution of 1st order eqs.

1st order eqs. are solved analytically and t-dependence is completely found

- Heat flux : $q^\mu(t, r, \theta, \varphi)$

$$
\begin{cases}
q^t = 0 \\
q^r = C_\mathrm{h}^r(r\,,\,\theta\,,\,\varphi)\,\exp\left(-\dfrac{\sqrt{A}}{\tau_\mathrm{h}}\,t\right) - \lambda\,\dfrac{B}{\sqrt{A}}\,\left(\sqrt{A}\,T\right)_{,r} \\
q^\theta = C_\mathrm{h}^\theta(r\,,\,\theta\,,\,\varphi)\,\exp\left(-\dfrac{\sqrt{A}}{\tau_\mathrm{h}}\,t\right) \\
q^\varphi = C_\mathrm{h}^\varphi(r\,,\,\theta\,,\,\varphi)\,\exp\left(-\dfrac{\sqrt{A}}{\tau_\mathrm{h}}\,t\right)
\end{cases}
$$

where $C_\mathrm{h}^\mu = \left(0\,,\,C_\mathrm{h}^r\,,\,C_\mathrm{h}^\theta\,,\,C_\mathrm{h}^\varphi\right)$: arbitrary function of (r, θ, φ)

$$
\rightarrow \quad \lim_{t\to\infty} q^r = -\lambda\,\frac{B}{\sqrt{A}}\,\left(\sqrt{A}\,T_{,r}\right)_{,r} \;\neq 0 \text{ even for } t > \frac{\tau_\mathrm{h}}{\sqrt{A}}
$$

- Bulk viscosity : $\Pi(t, r, \theta, \varphi) \;=\; C_{\rm b}(r\,,\,\theta\,,\,\varphi)\,\exp\Bigl(-\frac{\sqrt{A}}{\tau_{\rm b}}\,t\Bigr)$

  where, by the 1st order constraints $C_{\rm b} = \dfrac{\tilde{C}_{\rm b}(\theta, \varphi)}{\sqrt{A}}$

  $\rightarrow$ $\tilde{C}_{\rm b}$: arbitrary function of (θ, φ)

- Shear viscosity : $\overset{\circ}{\Pi}{}^{\mu\nu}(t, r, \theta, \varphi) \;=\; C_{\rm s}^{\mu\nu}(r\,,\,\theta\,,\,\varphi)\,\exp\Bigl(-\frac{\sqrt{A}}{\tau_{\rm s}}\,t\Bigr)$

  where $C_{\rm s}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & C_{\rm s}^{rr} & C_{\rm s}^{r\theta} & C_{\rm s}^{r\varphi} \\ & & C_{\rm s}^{\theta\theta} & C_{\rm s}^{\theta\varphi} \\ & \text{sym.} & & C_{\rm s}^{\varphi\varphi} \end{pmatrix}$: arbitrary function of (r, θ, φ)

$\rightarrow \displaystyle\lim_{t\to\infty} \Pi = 0$ and $\displaystyle\lim_{t\to\infty} \overset{\circ}{\Pi}{}^{\mu\nu} = 0$ $\cdots$ Viscosities vanish (relax) for $t > \dfrac{\tau_{\rm b,s}}{\sqrt{A}}$

- Velocity perturbation : $\delta u^\mu(t,r,\theta,\varphi) \to$ using the 1st order constraints :

$$
\begin{cases}
\delta u^t = 0 \\
\delta u^r = -\dfrac{C_{\rm h}^r}{\rho\,\varepsilon + p}\, e^{-(\sqrt{A}/\tau_{\rm h})\,t} + U \quad (\leftarrow \text{known function, see below}) \\
\delta u^\theta = \dfrac{1}{\rho\,\varepsilon + p}\Big[-C_{\rm h}^\theta\, e^{-(\sqrt{A}/\tau_{\rm h})\,t} + \dfrac{\tau_{\rm b}}{r^2}\, C_{{\rm b},r}\, e^{-(\sqrt{A}/\tau_{\rm b})\,t} \\
\qquad\qquad + \tau_{\rm s}\left\{\text{terms made of } C_{\rm s}^{\alpha\beta}\right\} e^{-(\sqrt{A}/\tau_{\rm s})\,t} \Big] + C_\delta{}^\theta \\
\delta u^\varphi = \dfrac{1}{\rho\,\varepsilon + p}\Big[-C_{\rm h}^\varphi\, e^{-(\sqrt{A}/\tau_{\rm h})\,t} + \dfrac{\tau_{\rm b}}{r^2}\, C_{{\rm b},r}\, e^{-(\sqrt{A}/\tau_{\rm b})\,t} \\
\qquad\qquad + \tau_{\rm s}\left\{\text{terms made of } C_{\rm s}^{\alpha\beta}\right\} e^{-(\sqrt{A}/\tau_{\rm s})\,t} \Big] + C_\delta{}^\varphi
\end{cases}
$$

where $C_\delta{}^\theta$, $C_\delta{}^\varphi$: arbitrary functions of (r,θ,φ)

$$
\to \lim_{t\to\infty} \delta u^r = U := \frac{\left[\lambda \frac{B}{\sqrt{A}}\left(\sqrt{A}\,T\right)_{,r}\right]_{,r} + \lambda\left(\frac{A_{,r}}{A} - \frac{R_{,r}}{2R} - \frac{2}{r}\right)\frac{B}{\sqrt{A}}\left(\sqrt{A}T\right)_{,r}}{\rho\left(\varepsilon_{,r} + p\,V_{,r}\right)} \neq 0
$$

- Constraints : Some differential relations of C_{h}^{μ} , $\tilde{C}_{\mathrm{b}}$, $C_{\mathrm{s}}^{\mu\nu}$ are given

  $\rightarrow$ does not affect the behavior of solutions for $t > \dfrac{\tau_{h,b,s}}{\sqrt{A}}$

- From the above $\cdots$

  Once dissipations arise, the star never relaxes to static state

$\rightarrow$ **The non-zero heat flux and velocity perturbation remain flowing**

$$\rightarrow \text{Is this imply} \begin{cases} \text{dissipative instability of hydrostatic star ?} \\ \text{the rise of a convection current ?} \end{cases}$$

5. Toward the BH Accretion : Present Status of Research

Consider spherical accretion onto Schwarzschild BH $\cdots$ a model of Sgr.A* ?

5.1 Zero-th order perfect flow : Bondi Flow

- Metric : $ds^2 = -\left(1 - \dfrac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

  $\rightarrow$ Eddington-Finkelstein Ingoing coordinates extend beyond BH horizon

- 0th order perfect fluid : **General Relativistic Bondi Flow**

  $\rightarrow$ $u^{\mu}_{(\mathrm{p})}(r) = \left(u^{v}_{(\mathrm{p})} , u^{r}_{(\mathrm{p})} , 0 , 0 \right)$, $\rho(r)$, $T(r)$: Given implicitly

  $\rightarrow$ Analysis of 1st order eqs. becomes difficult

5.2 1st order eqs.

$\rightarrow$ Numerical approach is under consideration

$\rightarrow$ Basic algorithm/strategy is almost constructed

6. Summary and Next Tasks

- Basic eqs. of EIT is too complicated

  $\rightarrow$ its concrete application to astrophysics has not been done

  $\rightarrow$ **I propose a perturbative approach to get benefit of EIT** :

  Expansion of u^μ around "Perfect Flow"

- An exercise : dissipative perturbation of Hydrostatic Star

  $\rightarrow$ I find that the dissipative perturbation never relax

  $\rightarrow$ There may some dissipative instability $\cdots$?

- First application to BH accretion : Dissipative Bondi Flow

  $\rightarrow$ Analytic approach is difficult

  $\rightarrow$ Basic strategy of numerical approach is almost constructed

  $\rightarrow$ **Next Task** is to complete it